



Princeton Computer Science Contest – Fall 2023

Problem 7: What's the point? (20 points) [File Upload]

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What is the point of life? What is the point of points? Are these questions meaningless or do they have a point? Honestly, at this point in the contest, you don't know and neither do we. So, we'll do the next best thing: solve some interesting problems about points. Hopefully you don't find this exercise pointless. Partial credit (some number of points) will be awarded for each part.

You are given N distinct points in the Cartesian plane. Observe that because two points define a line segment, a set of N distinct points defines $\binom{N}{2}$ line segments, each of which have length d_i . Because I am amazing at naming things, we call the *distance-line-ordering* of any set of N points to be an ordering of distances d_i such that $d_1 \leq d_2 \leq d_3 \cdots \leq d_{\binom{N}{2}}$.

Part 1

Show that for a set of N distinct points, the number of segments with distance $d_{\binom{N}{2}}$ is at most N . Is this upper bound tight? Prove your result.

Part 2

Show that for a set of N distinct points, the number of segments with distance d_1 is at most $3N$. Is the upper bound tight? Prove your result.

Part 3

Now, for a more pointed question. We call a subset of N points *unequivocally unequilateral* if no 3 points in the subset form an equilateral triangle. Show that every set of N distinct points contains an *unequivocally unequilateral* subset of size at least \sqrt{N} .

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